Dynamical evolution of escape probability in the presence of Sinai disorder

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The escape probability x_i from a site i on a Sinai lattice is treated as a discrete dynamical evolution by random iterations over two nonlinear maps. The maps arise in the context of a dichotomic model for Sinai diffusion. The global dynamics exhibits intermittent behavior with a long sojourn in the vicinity of a common fixed point of the maps. The intermittent dynamics is found to be nonchaotic. The intermittent behavior is characterized by a laminar interval, correlation function, and power spectrum.
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Models based on random walks to study anomalous diffusion in systems with quenched in disorder have received considerable attention over the past ten years [1–17]. Of particular interest has been the Sinai lattice [8–17], wherein the mean-square displacement of diffusing particle increases ultra slowly as the fourth power of the logarithm of time. Numerical studies [16] have revealed interesting fractal measures associated with the fluctuations of the probability distribution from one random realization of the Sinai lattice to the other. There have also been numerous studies reported [9–13] on the statistics of the first passage time from one end of a segment of size L of a Sinai lattice to the other. The mean first passage time (MFPT), averaged over the Sinai disorder, diverges with the system size as $\exp(L)$. The typical value, however, diverges rather slowly as $\exp(\sqrt{L})$, signaling the emergence of a long tail in the asymptotic distribution of the MFPT. A recent numerical study [17] has brought out the multifractal measures of the MFPT induced by the Sinai disorder. Recent studies [18] have shown an interesting possibility of casting the Sinai diffusion in the context of the iterated function system and it is the purpose of this paper to finish some of the unfinished tasks in this approach.

We consider a one-dimensional lattice, with the lattice sites labeled by the set of integers $\{i\}$. The particle at site i can jump to $i+1$ or $i-1$ with probabilities p_i and q_i, respectively. We have $p_i+q_i=1$. $\{p_i\}$ constitute a set of independent random variables with a common distribution that obeys the Sinai condition, namely, the quantity $\ln(q_i/p_i)$ is of zero mean and finite variance. We consider a binary model for the distribution and prescribe each p_i to take values of $\frac{1}{2}\pm\epsilon$, with equal probability, where $0<\epsilon<\frac{1}{2}$ measures the strength of the Sinai disorder.

Let $\hat{G}_{i,i+1}(n)$ denote the probability for the particle at site i to make a first passage to the site $i+1$ in n steps. $\hat{G}_{i,i+1}(n)$ obeys a master equation given by

$$\hat{G}_{i,i+1}(n)=q_i\hat{G}_{i-1,i+1}(n)+p_i\delta_{n,1}. \tag{1}$$

Let $G_{i,j}(z)$ denote the generating function corresponding to $\hat{G}_{i,j}(n)$, defined as

$$G_{i,j}(z)=\sum_{n=1}^{\infty} z^n\hat{G}_{i,j}(n). \tag{2}$$

Accordingly, Eq. (1) can be transformed to yield

$$G_{i,i+1}(z)=\frac{zp_i}{1-zq_iG_{i-1,i}(z)}, \tag{3}$$

where we have made use of the convolution $G_{i-1,i+1}(z)=G_{i-1,i}(z)G_{i,i+1}(z)$. If we substitute $z=1$ into Eq. (3) we get the total probability for the particle to make a first passage from i to $i+1$ in terms of that of a first passage for $i-1$ to i. We denote by $x_i=G_{i,i+1}(z=1)$ the escape probability from site i to $i+1$. We take x_0 to be in the open interval [0,1]. Equation (3), with $z=1$, provides a recursion for x_i in terms of x_{i-1}, given x_0. We can interpret this as a dynamical evolution of an initial probability x_0 as

$$x_i=\eta_i f(x_{i-1})+(1-\eta_i)g(x_{i-1}), \tag{4}$$

where the two nonlinear maps are given by

$$f(x)=\frac{p}{1-qx}, \tag{5a}$$

$$g(x)=\frac{q}{1-px}, \tag{5b}$$

with the Sinai prescription that at each stage of iteration the map f or g is chosen independently and randomly with equal probability. This is accomplished by prescribing $\{\eta_i\}$ in Eq. (4) to be a set of identically distributed independent random variables with each η_i taking a value of zero or one with equal probability. Without loss of generality we take $p=\frac{1}{2}+\epsilon$ and $q=\frac{1}{2}-\epsilon$, so that $p>q$. Equations (4) and (5) constitute an iterated function system (IFS) (see [19]) and is denoted by $\mathcal{F}$.

In the range 0–1, the map f has a single fixed point at $x=y_2=1$. Calculating the first derivative of $f(x)$ with respect to x at 1, we get $f'(x=1)=q/p<1$, which shows that the fixed point is stable. Also, the map is contracting since $f'(x)<1$ for all $0<x<1$. The second map g has two fixed points, one at $x=y_1=(1-\sqrt{1-4pq})/2p$ and the other at $x=y_2=1$. It is easily verified that $g'(x=y_1)<1$ and hence the fixed point at 1 is unstable. The map $g(x)$ is also contracting except in a region $(1-\sqrt{pq})/p<x\leqslant 1$, where the slope

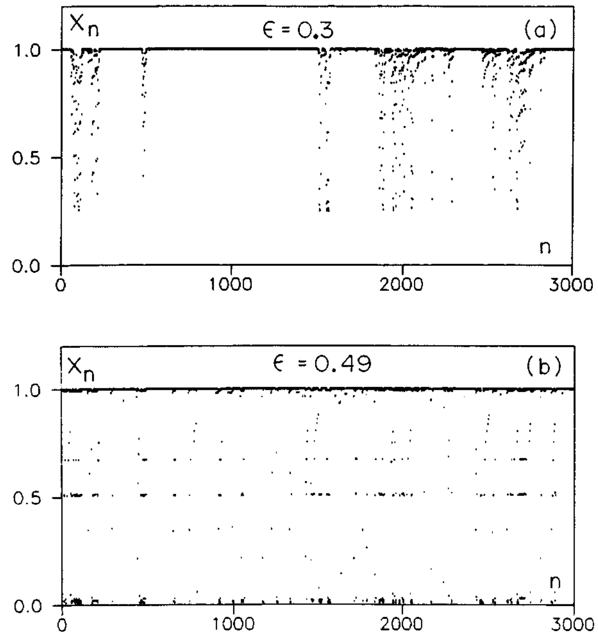

FIG. 1. Sample trajectories for the Sinai problem for (a) $\epsilon=0.3$ and (b) $\epsilon=0.49$ and with an infinitesimal absorption $\gamma=10^{-14}$.

$g'(x)$ is greater than unity. The asymptotic global dynamics is confined to the interval (y_1,y_2), constituting the compact metric support for the IFS $\mathcal{F}$. Though each of the two maps has one distinct and stable fixed point, random iterations render the trajectory wander endlessly in the interval $[y_1,1]$ without ever letting it reach the fixed points.

We carry out Monte Carlo simulation of Eq. (4) by selecting first x_0 from a uniform distribution between y_1 and y_2. We then calculate $x_1,x_2,\ldots,$ iteratively employing a string of random numbers $\{\eta_i\}$. All the calculations reported here have been performed in standard double precision arithmetic. Figure 1 depict sample dynamical trajectories for $\epsilon=0.3$ and 0.49. The trajectory exhibits intermittent behavior [20–22], with a long sojourn in the vicinity of unity for $\epsilon=0.3$. The plot of x_n versus n in Fig. 1(a) for $\epsilon=0.3$ indicates that the laminar period is 1. When ϵ is increased the laminar region decreases and the duration of irregular bursts increases. The evolution becomes highly irregular for $\epsilon=0.49$ [Fig. 1(b)]. To understand the intermittent behavior let us look at the nature of the dynamics in the vicinity of the upper fixed point at $x=1$. This can be done analytically as follows.

Let $x_N(x_0)$ denote the value of x after N random iterations over the maps f and g starting from x_0. There is a total of 2^N possible dynamical trajectories emanating from x_0, leading to different values of $x_N(x_0)$. Let us consider trajectories emanating from a point infinitesimally close to x_0 and get $x_N(x_0+\xi_0)$. Let $\xi(N)=x_N(x_0+\xi_0)-x_N(x_0)$. We have

$$\xi(N)=\xi_0 S(x_0)+O(\xi_0^2), \tag{6}$$

where $S(x_0)$ denotes the first derivative of $x_N(x_0)$ with respect to x_0. We are interested in investigating the nature of dynamical system in the vicinity of $x_0=1$. We have $f'(1)$

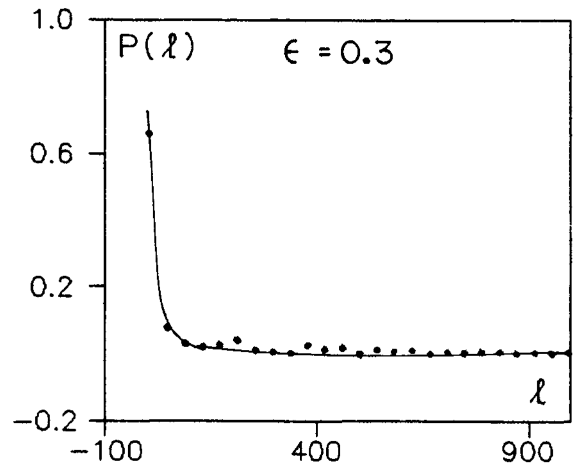

FIG. 2. Laminar interval l versus its probability distribution $P(l)$ for $\epsilon=0.3$.

$=q/p$ and $g'(1)=p/q$. Using the fact that $x_0=1$ is a common fixed point for both f and g, we get

$$S(x_0=1)=(p/q)^{N-2L}, \tag{7}$$

where L is a random variable having a binomial distribution with mean $N/2$. The average value of $S(x_0=1)$ is given by

$$\langle S(x_0=1)\rangle=\left[\frac{p/q+q/p}{2}\right]^N. \tag{8}$$

Thus an initial difference of ξ_0 increases on the average exponentially with N and the exponent characterizing this divergence is given by

$$\lambda_1(x_0)=\ln\left[\frac{p/q+q/p}{2}\right]>0. \tag{9}$$

On the other hand, if we first take the logarithm of the slope $S(x_0=1)$ and then take the average, we get

$$\lambda_2(x_0)=\left\langle\frac{\ln[S(x_0=1)]}{N}\right\rangle=\left\langle\frac{N-2L}{N}\right\rangle\ln(p/q)=0, \tag{10}$$

indicating that the dynamics is marginally stable in the vicinity of the upper fixed point, at 1. We contend that the exponent λ_1 describes the stability of the average dynamics and the exponent λ_2 describes that of a typical dynamical trajectory. The picture that emerges from the above discussion is that the dynamical trajectory spends considerable time in the vicinity of the upper fixed point since $\lambda_2=0$; however, in the long run, the trajectory is bound to move out since $\lambda_1>0$. After spending a relatively short time away, the trajectory returns to the vicinity of the upper fixed point. This process repeats and we get intermittent behavior, as shown in Figs. 1(a) and 1(b).

Recently, Loreto *et al.* [23] studied a logistic map by considering the Perron-Frobenius operator where the control parameter is switched randomly into one of the two chosen values. In the map (4) the random choice of f and g plays the role of stochastic noise source.

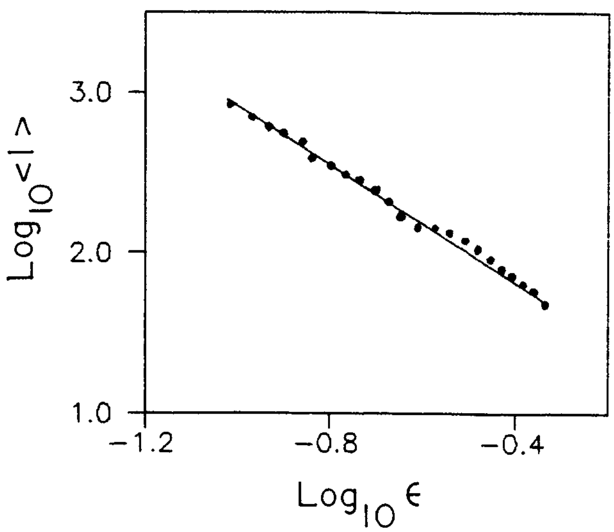

FIG. 3. The dots show the average laminar interval $\langle l \rangle$ and the solid line is the best straight line fit.

To analyze the intermittency behavior we studied scaling of an average laminar interval with ϵ, probability function, and power spectrum. In the intermittency region, the time series consists of regular laminar motion interrupted by irregular bursts. The laminar motion between successive bursts has different durations that are randomly distributed over the time series. The set $(x_{i+1}, x_{i+2}, \ldots, x_{i+n})$ is called the laminar interval [21] if $|x_i - x_c|$ and $|x_{i+n+1} - x_c|$ are both greater than a preassumed gate value δ and $|x_{i+j} - x_c| < \delta$, $j = 1,2,...,n$, where x_c is the fixed point around which slow passage of the iterates occurs. In the present analysis $x_c = 1$ and δ is fixed as 10^{-2}. In Fig. 2 we plotted the probability distribution $P(l)$ for $\epsilon = 0.3$. We found $P(l) \approx 3.66 \exp(-3.42\epsilon l)$. The variation of average laminar length $\langle l \rangle$ with ϵ is also studied. For a given ϵ value the map is iterated and 5000 laminar intervals l_i are calculated. Then the average laminar length is obtained. Figure 3 depicts the variation of the average number $\langle l \rangle$ with ϵ on a $\log_{10}$-$\log_{10}$ plot. The dots are the numerical result and continuous line is the least-squares straight-line fit. The calculated $\langle l \rangle$ scales as $\langle l \rangle \approx 11.84\epsilon^{-1.85}$. For the map (4) one expects [21] $\langle l \rangle$ to scale as $\sim \epsilon^{-2}$.

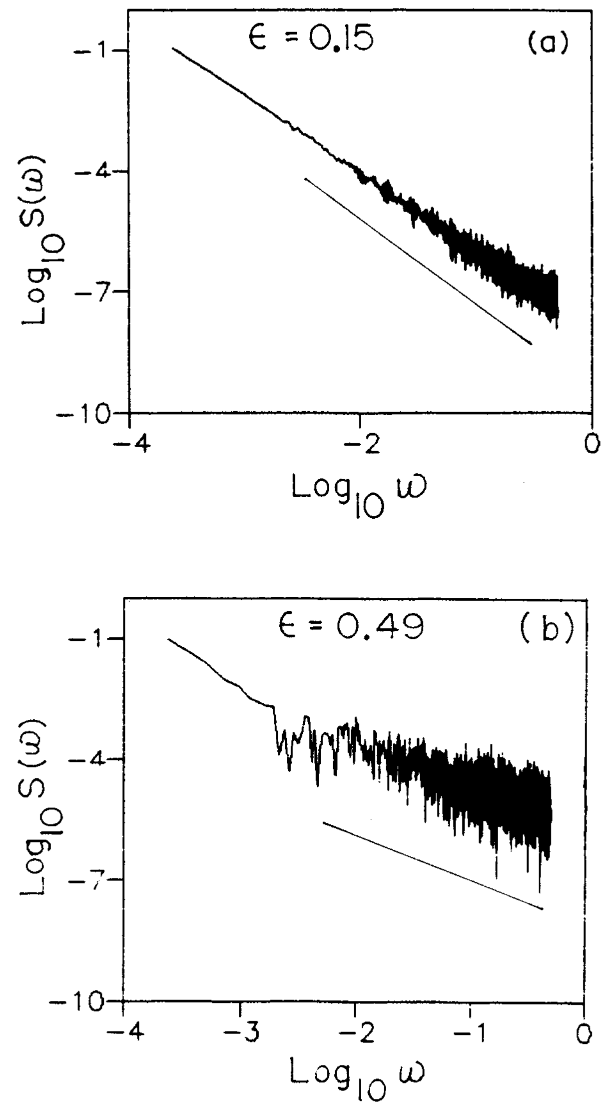

FIG. 5. Power spectrum of trajectories for (a) $\epsilon = 0.15$ showing a $1/w^2$ dependence and (b) $\epsilon = 0.49$ showing a $1/w$ dependence.

The autocorrelation function

$$C(\tau) = (1/N) \sum_{i=1}^{N} (x_i - \langle x \rangle)(x_{i+\tau} - \langle x \rangle),$$

where

$$\langle x \rangle = (1/N) \sum_{j=1}^{N} x_j,$$

obtained for two values of ϵ, are shown in Fig. 4. These were normalized so that $C(0) = 1$. The continuous and dotted curves correspond to $\epsilon = 0.15$ and 0.49, respectively. From Fig. 4 it is clear that as ϵ increases the correlation length characteristic of the decay decreases.

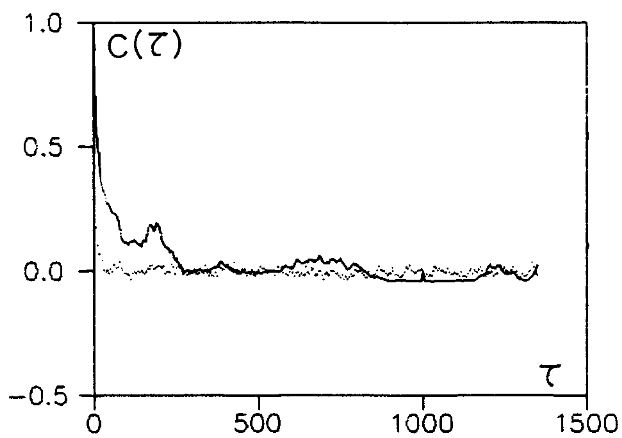

FIG. 4. Correlation function of the map (4). The continuous curve and dots corresponds to $\epsilon = 0.15$ and 0.49, respectively.

A power spectrum is also computed for two ϵ values, namely, $\epsilon = 0.15$ and 0.49. For $\epsilon = 0.15$ bursts occur occasionally, whereas for $\epsilon = 0.49$ irregular bursts are found to occur very often. A different scaling behavior of the power spectrum $S(w)$ is found for these two distinct evolutions. The numerically computed power spectrum is displayed in Fig. 5. For $\epsilon = 0.15$ the power spectrum has an approximate $1/w^2$ dependence. On the other hand, for $\epsilon = 0.49$ the power spectrum scales as $S(w) \sim 1/w$.

In summary, we have shown that anomalous diffusion on a Sinai lattice can be studied within the framework of an iterated function system. The dynamics exhibits an extrinsic type of intermittency.

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[1] J. W. Haus and K. W. Kehr, Phys. Rep. **150**, 263 (1987); S. Havlin and D. Ben-Avraham, Adv. Phys. **36**, 695 (1985).
[2] S. Alexander, J. Bernasconi, W. R. Schneider, and R. Orbach, Rev. Mod. Phys. **53**, 175 (1981); J. P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990).
[3] S. Aslangul, J. P. Bouchaud, A. Georges, N. Potier, and D. S. James, J. Stat. Phys. **55**, 461 (1989).
[4] M. Raykin, J. Phys. A **26**, 449 (1993).
[5] P. Pury, M. O. Caceres, and E. H. Garcia, Phys. Rev. E **49**, R967 (1994).
[6] S. Havlin, R. B. Selinger, M. Schwartz, H. E. Stanley, and A. Bunde, Phys. Rev. Lett. **61**, 1438 (1988).
[7] C. van den Broeck, in *Proceeding of the NATO Conference on Noise and Nonlinear Phenomena in Nuclear Systems*, edited by J. L. Munoz Cobo and H. F. C. Difilipo (Plenum, New York, 1988).
[8] Ya. G. Sinai, Theor. Probab. Appl. **27**, 247 (1982).
[9] S. H. Noskowicz and I. Goldhirsch, Phys. Rev. Lett. **61**, 500 (1988).
[10] K. P. N. Murthy and K. W. Kehr, Phys. Rev. A **40**, 2082 (1989); **41**, 1160 (1990).
[11] P. Le Doussal, Phys. Rev. Lett. **62**, 3097 (1989).
[12] K. P. N. Murthy and K. W. Kehr, Phys. Rev. A **41**, 5728 (1990).
[13] S. H. Noskowicz and I. Goldhirsch, Phys. Rev. A **42**, 2047 (1990).
[14] C. Aslangul, N. Potier, and D. Saint-James, Physica A **164**, 52 (1990).
[15] R. B. L. Selinger, S. Havlin, F. Leyvraz, M. Schwartz, and H. E. Stanley, Phys. Rev. A **40**, 6755 (1989).
[16] H. E. Roman, A. Bunde, and S. Havlin, Phys. Rev. A **38**, 2185 (1988).
[17] K. P. N. Murthy, S. Rajasekar, and K. W. Kehr, J. Phys. A **27**, L112 (1994).
[18] C. van den Broeck, J. Stat. Phys. **65**, 971 (1991); Physica A **188**, 452 (1992).
[19] M. F. Barnsley, *Fractals Everywhere* (Academic, New York, 1988).
[20] P. Szepfalusy and G. Gyorgyi, Phys. Rev. A **33**, 2852 (1986); T. Geisel and S. Thomae, Phys. Rev. Lett. **52**, 1936 (1984); Y. Pomeau and P. Manneville, Commun. Math. Phys. **74**, 189 (1980).
[21] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Proccacia, and B. Shraiman, Phys. Rev. A **33**, 1141 (1986).
[22] E. G. Gwinn and R. M. Westervelt, Phys. Rev. A **33**, 4143 (1986).
[23] V. Loreto, G. Paladin, M. Pasquini, and A. Vulpiani, Physica A **232**, 189 (1996).